Note di Matematica
ISSN 1123-2536, e-ISSN 1590-0932
Note Mat. **32** (2012) n. 1, 159–173.
doi:10.1285/i15900932v32n1p159

Onset of convection in porous layers salted from above and below

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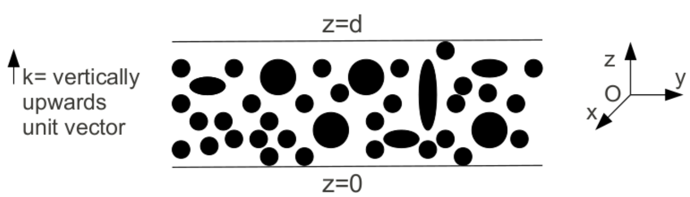
Abstract. For a porous layer heated from below and salted from above and below, the non existence of subcritical instabilities and conditions of global stability - for special values of the Prandtl numbers - are found.

Keywords: Multi-component fluid mixtures, Porous media, Convection, Stability.

1 Introduction and aims

A porous medium is schematized via a body (generally rigid and called *skeleton*) having interconnected pores everywhere. Generally, the fluid occupying the pores is a mixture since are dissolved in chemical species ("salts") and the layer is embedded in a temperature field.

The behaviour of convective-diffusive fluid mixtures in porous layer presents a picture of behaviours increasing with the number of components. Although the



subject of double-diffusive convection is still a very active research area [1]-[22], the same subject with more than two components, although more difficult – in the past as nowadays has also attracted the attention of many authors [23]-[30].

The present paper is concerned with an horizontal layer heated from below and salted by two salts, either from below ("salt 1") or from above ("salt 2").

Denoting by

$$\begin{cases} T = \text{temperature field}, \quad C_i = \text{concentration field of the "salt } i\text{"}, (i = 1, 2) \\ \\ \mathbf{v} = \text{ seepage velocity}, \quad p = \text{ pressure field}, \end{cases} \tag{1}$$

in the case of the boundary conditions

$$\begin{cases} T(0) = T_1, \quad T(d) = T_2, \quad T_1 - T_2 > 0, \\ \\ C_i(0) = C_{il}, \quad C_i(d) = C_{iu}, \quad (i = 1, 2), \\ \\ \mathbf{v} \cdot \mathbf{k} = 0, \quad \text{at } z = 0, \ z = d, \end{cases} \tag{2}$$

it follows that the perturbations $(\mathbf{u}, \Phi_1, \Phi_2, \Pi)$ to the conduction solution {cfr. [29]}, with

$$\begin{cases} \mathbf{u} = \text{ perturbation to } \mathbf{v}, \\ \Phi_i = \text{ perturbation to } C_i, \\ \Pi = \text{ perturbation to } p, \end{cases} \tag{3}$$

are governed by

$$\begin{cases} \nabla \Pi = -\mathbf{u} + (R\theta - R_1\Phi_1 - R_2\Phi_2)\mathbf{k}, \\ \\ \nabla \cdot \mathbf{u} = 0, \\ \\ \theta_t + \mathbf{u} \cdot \nabla\theta = R\omega + \Delta\theta, \\ \\ P_1(\Phi_{1t} + \mathbf{u} \cdot \nabla\Phi_1) = R_1\omega + \Delta\Phi_1, \\ \\ P_2(\Phi_{2t} + \mathbf{u} \cdot \nabla\Phi_2) = -R_2\omega + \Delta\Phi_2, \end{cases} \tag{4}$$

$$\omega = \theta = \Phi_1 = \Phi_2 = 0, \qquad \text{on } z = 0, z = 1, \tag{5}$$

with

$$\begin{cases} \omega = \mathbf{u} \cdot \mathbf{k}, \\ R = \text{Rayleigh thermal number}, \\ R_i = \text{Rayleigh concentration number of "salt } i\text{"}. \end{cases} \tag{6}$$

Assuming - as it is normally done - that

i) $\mathbf{u} = (u, v, \omega), \theta, \Phi_1, \Phi_2$ are periodic in the x and y directions respectively of periods $2\pi/a_x$, $2\pi/a_y$;

ii) $\Omega = [0, 2\pi/a_x] \times [0, 2\pi/a_y] \times [0, 1]$ is the periodicity cell;

iii) $\mathbf{u}, \theta, \Phi_1, \Phi_2$ belong to $W^{2,2}(\Omega)$ and are such that all their first derivatives and second spatial derivatives can be expanded in Fourier series uniformly convergent in Ω,

our aim - according to the results obtained in {[29]-[30],[33] } - is to show that when the layer is salted from below by "salt 1" and from above by "salt 2", then

1) do not exist subcritical instabilities;

2) exist physically relevant values of the "salts" Prandtl numbers such that the triply diffusive-convection can be reduced rigorously to the double diffusive-convection and the global stability condition is given by $R^2 < R_C^2$ with

$$\begin{cases} R_c^2 = \min\left[R_1^2 - \dfrac{R_2^2}{P_2} + 4\pi^2\left(1 + \dfrac{1}{P_2}\right), R_1^2 - R_2^2 + 4\pi^2\right], \\ \text{when } P_1 = 1 \end{cases} \tag{7}$$

$$\begin{cases} R_c^2 = \min\left[\dfrac{R_1^2}{P_1} - R_2^2 + 4\pi^2\left(1 + \dfrac{1}{P_1}\right), R_1^2 - R_2^2 + 4\pi^2\right], \\ \text{when } P_2 = 1, \end{cases} \tag{8}$$

$$\begin{cases} R_c^2 = \min\left[\dfrac{1}{P}(R_1^2 - R_2^2) + 4\pi^2\left(1 + \dfrac{1}{P}\right), R_1^2 - R_2^2 + 4\pi^2\right], \\ \text{when } P_1 = P_2 = P. \end{cases} \tag{9}$$

Section 2 is devoted to the boundary value problem of the problem at stake. The (Routh Hurwitz) conditions of linear stability are found in the subsequent Section while Sections 4-5 are devoted to the non existence of subcritical instabilities. Finally, (7)-(9) are shown in Section 6.

2 The boundary value problem at stake

We recall here a basic theorem - which proof is given either in [29] or [31] - concerned with the main boundary value problem $(4)_1$-$(4)_2$-(5).

Theorem 1. *Let* $(\mathbf{u}, \theta, \Phi_1, \Phi_2)$ *be solution of the b.v.p.*

$$\begin{cases} \nabla\Pi = -\mathbf{u} + (R\theta - R_1\Phi_1 - R_2\Phi_2)\mathbf{k}, \\ \nabla\cdot\mathbf{u} = 0, \end{cases} \tag{10}$$

$$\omega = \theta = \Phi_1 = \Phi_2 = 0, \qquad z = 0, 1, \tag{11}$$

then:

i) $(\mathbf{u}, \theta, \Phi_1, \Phi_2)$ *is solution of the b.v.p.*

$$\begin{cases} \Delta\omega = \Delta_1(R\theta - R_1\Phi_1 - R_2\Phi_2), \quad in\ \Omega, \\ \mathbf{u} = \theta = \Phi_1 = \Phi_2 = 0, \qquad on\ z = 0, 1, \\ \Delta_1 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}. \end{cases} \tag{12}$$

ii) a complete orthogonal system of solutions of (10) is given by

$$\begin{cases} \tilde{\omega}_n = \eta_n(R\tilde{\theta}_n - R_1\tilde{\Phi}_{1n} - R_2\tilde{\Phi}_{2n}), \\ \mathbf{u}_n = \dfrac{1}{a^2}\left(\dfrac{\partial^2\tilde{\omega}_n}{\partial x\partial z}\mathbf{i} + \dfrac{\partial^2\tilde{\omega}_n}{\partial y\partial z}\mathbf{j}\right) + \tilde{\omega}_n\mathbf{k}, \end{cases} \tag{13}$$

with

$$\begin{cases} a^2 = a_x^2 + a_y^2,\ \xi_n = a^2 + n^2\pi^2,\ \eta_n = \dfrac{a^2}{\xi_n}, \\ \omega = \displaystyle\sum_1^\infty \tilde{\omega}_n = \sum_1^\infty \omega_n(x, y, t)\sin(n\pi z), \\ \mathbf{u} = \displaystyle\sum_1^\infty \mathbf{u}_n = \sum_1^\infty (\tilde{u}_n\mathbf{i} + \tilde{v}_n\mathbf{j} + \tilde{\omega}_n\mathbf{k}), \\ \theta = \displaystyle\sum_1^\infty \tilde{\theta}_n = \sum_1^\infty \theta_n(x, y, t)\sin(n\pi z), \\ \Phi_i = \displaystyle\sum_1^\infty \tilde{\Phi}_{in} = \sum_1^\infty \Phi_{in}(x, y, t)\sin(n\pi z). \end{cases} \tag{14}$$

Remark 1. By virtue of theorem 1 the independent scalar unknown fields $(u, v, \omega, \theta, \Phi_1, \Phi_2)$ are reduced to (θ, Φ_1, Φ_2).

3 Linear instability

Linearizing (4) and taking into account that by virtue of theorem 1, it turns out that

$$\Delta\varphi = -\sum_1^\infty \xi_n\varphi_n, \qquad \varphi \in (\theta, \Phi_1, \Phi_2), \tag{15}$$

one obtains

$$\begin{cases} \theta_t = \sum_{n=1}^{\infty}(a_{1n}\theta_n + a_{2n}\Phi_{1n} + a_{3n}\Phi_{2n}), \\ \Phi_{1t} = \sum_{n=1}^{\infty}(b_{1n}\theta_n + b_{2n}\Phi_{1n} + b_{3n}\Phi_{2n}), \\ \Phi_{2t} = \sum_{n=1}^{\infty}(c_{1n}\theta_n + c_{2n}\Phi_{1n} + c_{3n}\Phi_{2n}), \end{cases} \tag{16}$$

with

$$\begin{cases} a_{1n} = R^2\eta_n - \xi_n, & a_{2n} = -RR_1\eta_n, & a_{3n} = -RR_2\eta_n, \\ b_{1n} = \dfrac{RR_1}{P_1}\eta_n, & b_{2n} = -\dfrac{(R_1^2\eta_n + \xi_n)}{P_1}, & b_{3n} = -\dfrac{R_1R_2}{P_1}\eta_n, \\ c_{1n} = -\dfrac{RR_2}{P_2}\eta_n, & c_{2n} = \dfrac{R_1R_2}{P_2}\eta_n, & c_{3n} = \dfrac{R_2^2\eta_n - \xi_n}{P_2}. \end{cases} \tag{17}$$

Setting

$$\begin{cases} \theta_n = \xi_{0n}(t)F(x,y)\sin(n\pi z), \\ \Phi_{in} = \xi_{in}(t)F(x,y)\sin(n\pi z), \end{cases} \tag{18}$$

by virtue of

$$\int_0^1 \sin(n\pi z)\sin(m\pi z)d\,z = \begin{cases} 0, & n \neq m \\ \dfrac{1}{2}, & n = m \end{cases} \tag{19}$$

(16) implies

$$\begin{cases} \dfrac{d\xi_{0n}}{dt} = a_{1n}\xi_{0n} + a_{2n}\xi_{1n} + a_{3n}\xi_{2n}, \\ \dfrac{d\xi_{1n}}{dt} = b_{1n}\xi_{0n} + b_{2n}\xi_{1n} + b_{3n}\xi_{2n}, \\ \dfrac{d\xi_{2n}}{dt} = c_{1n}\xi_{0n} + c_{2n}\xi_{1n} + c_{3n}\xi_{2n}. \end{cases} \tag{20}$$

Setting

$$\mathcal{L}_n = \begin{pmatrix} a_{1n} & a_{2n} & a_{3n} \\ b_{1n} & b_{2n} & b_{3n} \\ c_{1n} & c_{2n} & c_{3n} \end{pmatrix}$$

it easily follows that the characteristic equation of the $\mathcal{L}_n$ eigenvalues λ_n is

$$\lambda_n^3 - \mathtt{I}_{1n}\lambda_n^2 + \mathtt{I}_{2n}\lambda_n - \mathtt{I}_{3n} = 0, \tag{21}$$

with $\mathtt{I}_{1n}, \mathtt{I}_{2n}, \mathtt{I}_{3n}$ given by

$$\begin{cases} \mathtt{I}_{1n} = a_{1n} + b_{2n} + c_{3n} = \lambda_{1n} + \lambda_{2n} + \lambda_{3n}, \\[2mm] \mathtt{I}_{2n} = \begin{vmatrix} a_{1n} & a_{2n} \\ b_{1n} & b_{2n} \end{vmatrix} + \begin{vmatrix} a_{1n} & a_{3n} \\ c_{1n} & c_{3n} \end{vmatrix} + \begin{vmatrix} b_{2n} & b_{3n} \\ c_{2n} & c_{3n} \end{vmatrix}, \\[2mm] \mathtt{I}_{3n} = \begin{vmatrix} a_{1n} & a_{2n} & a_{3n} \\ b_{1n} & b_{2n} & b_{3n} \\ c_{1n} & c_{2n} & c_{3n} \end{vmatrix}. \end{cases} \tag{22}$$

By virtue of the Routh-Hurwitz stability-conditions [31], the following theorem holds.

Theorem 2. *The conduction solution is linearly stable if and only if, $\forall n \in \mathbb{N}$, the inequalities*

$$I_{1n} < 0,\ I_{3n} < 0,\ I_{1n}I_{2n} - I_{3n} < 0, \tag{23}$$

hold.

Since (23) requires $\mathtt{I}_{2n} > 0$, in view of

$$\begin{cases} \mathtt{I}_{1n} = \left[R^2 - \dfrac{R_1^2}{P_1} + \dfrac{R_2^2}{P_2} - \left(1 + \dfrac{1}{P_1} + \dfrac{1}{P_2}\right)\dfrac{\xi_n}{\eta_n}\right]\eta_n, \\[2mm] \mathtt{I}_{2n} = \left[\left(\dfrac{1}{P_1} + \dfrac{1}{P_2} + \dfrac{1}{P_1P_2}\right)\dfrac{\xi_n}{\eta_n} + \dfrac{1}{P_1}\left(1 + \dfrac{1}{P_2}\right)R_1^2 - \dfrac{1}{P_2}\left(1 + \dfrac{1}{P_1}\right)R_2^2 + \right. \\[2mm] \qquad\qquad \left. - \left(\dfrac{1}{P_1} + \dfrac{1}{P_2}\right)R^2\right]\xi_n\eta_n, \\[2mm] \mathtt{I}_{3n} = \dfrac{1}{P_1P_2}\left(R^2 - R_1^2 + R_2^2 - \dfrac{\xi_n}{\eta_n}\right)\eta_n\xi_n^2, \\[2mm] \inf\limits_{(a^2,n)\in\mathbb{R}^+\times\mathbb{N}} \dfrac{\xi_n}{\eta_n} = 4\pi^2, \end{cases} \tag{24}$$

it follows that

Theorem 3. *The conduction solution is linearly stable only if*

$$R_{C_i}^2 > 0,\ (i = 1, 2, 3), \quad R^2 < R_C^2, \tag{25}$$

with

$$\begin{cases} R^2_{C_1} = \dfrac{R_1^2}{P_1} - \dfrac{R_2^2}{P_2} + 4\pi^2\left(1 + \dfrac{1}{P_1} + \dfrac{1}{P_2}\right), \\ \\ R^2_{C_2} = \dfrac{1}{P_1 + P_2}\left[(1+P_2)R_1^2 - (1+P_1)R_2^2 + 4\pi^2(1 + P_1 + P_2)\right], \\ \\ R^2_{C_3} = R_1^2 - R_2^2 + 4\pi^2,\ R^2_C = \min(R^2_{C_1}, R^2_{C_2}, R^2_{C_3}). \end{cases} \tag{26}$$

4 Preliminaries to nonexistence of subcritical instabilities

Lemma 1. *Let λ_1 be a real eigenvalue of the matrix*

$$L = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}, \qquad \begin{pmatrix} \alpha_{ij} = const \in \mathbb{R}) \\ i, j = 1, 2, 3 \end{pmatrix} \tag{27}$$

and let $\tilde{\mathbf{U}} = (1, \tilde{U}_2, \tilde{U}_3)$ be an associated eigenvector. Then the transformation

$$\mathbf{X} = L_1\mathbf{Z}, \tag{28}$$

with

$$\mathbf{X} = (X_1, X_2, X_3)^T,\ \mathbf{Z} = (Z_1, Z_2, Z_3)^T,\ L_1 = \begin{pmatrix} 1 & 0 & 0 \\ \tilde{U}_2 & 1 & 0 \\ \tilde{U}_3 & 0 & 1 \end{pmatrix} \tag{29}$$

reduces the ternary system

$$\frac{d\mathbf{X}}{dt} = L\mathbf{X} + \mathbf{F}, \tag{30}$$

with $(\mathbf{F})_{\mathbf{X}=0} = 0$, to

$$\frac{d\mathbf{Z}}{dt} = \tilde{L}\mathbf{Z} + \tilde{\mathbf{F}}, \tag{31}$$

$$\tilde{L} = \begin{pmatrix} \lambda_1 & \tilde{\alpha}_{12} & \tilde{\alpha}_{13} \\ 0 & \tilde{\alpha}_{22} & \tilde{\alpha}_{23} \\ 0 & \tilde{\alpha}_{32} & \tilde{\alpha}_{33} \end{pmatrix}, \tilde{\mathbf{F}} = (\tilde{F}_1, \tilde{F}_2, \tilde{F}_3)^T. \tag{32}$$

$$\begin{cases} Z_1 = X_1, Z_2 = X_2 - \tilde{U}_2X_1, Z_3 = X_3 - \tilde{U}_3X_1, \\ \tilde{F}_1 = F, \tilde{F}_2 = F_2 - \tilde{U}_2F_1, \tilde{F}_3 = F_3 - \tilde{U}_3F_1 \end{cases} \tag{33}$$

$$\begin{cases} \tilde{\alpha}_{12} = \alpha_{12}, \ \tilde{\alpha}_{13} = \alpha_{13}, \\ \tilde{\alpha}_{22} = \alpha_{22} - \tilde{U}_2\alpha_{12}, \ \tilde{\alpha}_{23} = \alpha_{23} - \tilde{U}_2\alpha_{13}, \\ \tilde{\alpha}_{32} = \alpha_{32} - \tilde{U}_3\alpha_{12}, \ \tilde{\alpha}_{33} = \alpha_{33} - \tilde{U}_3\alpha_{13} \end{cases} \tag{34}$$

$$\tilde{\alpha}_{22} + \tilde{\alpha}_{33} = \lambda_2 + \lambda_3, \ \tilde{\alpha}_{22}\tilde{\alpha}_{33} - \tilde{\alpha}_{23}\tilde{\alpha}_{32} = \lambda_2\lambda_3, \tag{35}$$

λ_2, λ_3 being other eigenvalues of L.

Proof. The proof, based on [32, pp.194-197], can be found in [30], [33].

Lemma 2. *Let the eigenvalues λ_i, $(i = 1, 2, 3)$ of*

$$L = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \tag{36}$$

have negative real part and let $\lambda_1 < 0$ and

$$\Psi = (\tilde{A}_1 Z_2 - \tilde{A}_3 Z_3)\tilde{F}_2 + (\tilde{A}_2 Z_3 - \tilde{A}_3 Z_2)\tilde{F}_3 + Z_1\tilde{F}_1 = 0, \tag{37}$$

with

$$\begin{cases} \tilde{A}_1 = \lambda_2\lambda_3 + \tilde{\alpha}_{32}^2 + \tilde{\alpha}_{33}^2, \quad \tilde{A}_2 = \lambda_2\lambda_3 + \tilde{\alpha}_{22}^2 + \tilde{\alpha}_{23}^2, \\ \\ \tilde{A}_3 = \tilde{\alpha}_{22}\tilde{\alpha}_{32} + \tilde{\alpha}_{23}\tilde{\alpha}_{33}. \end{cases} \tag{38}$$

Then the function

$$\tilde{W} = \frac{1}{2}\left[Z_1^2 + \lambda_2\lambda_3(Z_2^2 + Z_3^2) + (\tilde{\alpha}_{22}Z_3 - \tilde{\alpha}_{32}Z_2)^2 + (\tilde{\alpha}_{23}Z_3 - \tilde{\alpha}_{33}Z_2)^2\right], \tag{39}$$

has - along

$$\frac{d\mathbf{Z}}{dt} = \tilde{L}Z + \tilde{\mathbf{F}}, \tag{40}$$

the temporal derivative given by

$$\frac{d\tilde{W}}{dt} = \frac{1}{2}\left[\lambda_1 Z_1^2 + (\lambda_2 + \lambda_3)\lambda_2\lambda_3(Z_2^2 + Z_3^2)\right] < 0. \tag{41}$$

The null solution of (40) and hence of (30) is globally stable and subcritical instabilities do not exist.

Proof. A detailed proof can be found in [30], [33].

Remark 2. The eigenvalues of the matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix}$$

have negative real part if and only if

$$a_{11} < 0,\ I = a_{22} + a_{33} < 0,\ A = a_{22}a_{33} - a_{32}a_{23}.$$

(cfr. Remark 2.5 of [30]).

5 Nonexistence of subcritical instabilities and global stability

Following [33], we set

$$S_m^{(\theta)} = \sum_{n=1}^{m} \theta_n,\ S_m^{(\Phi_i)} = \sum_{n=1}^{m} \Phi_{in},\ \mathbf{U}_m = \sum_{n=1}^{m} \mathbf{u}_n, \tag{42}$$

$$\theta = \lim_{m\to\infty} S_m^{(\theta)},\ \Phi_i = \lim_{m\to\infty} S_m^{(\Phi_i)},\ (i = 1, 2). \tag{43}$$

The nonexistence of subcritical instabilities and the global stability is guaranteed by showing that the asymptotic stability of the null solution of

$$\begin{cases} \dfrac{d}{dt} S_m^{(\theta)} = \displaystyle\sum_{n=1}^{m} (a_{1n}\theta_n + a_{2n}\Phi_{1n} + a_{3n}\Phi_{2n}) - \mathbf{U}_m \cdot \nabla S_m^{(\theta)}, \\ \dfrac{d}{dt} S_m^{(\Phi_1)} = \displaystyle\sum_{n=1}^{m} (b_{1n}\theta_n + b_{2n}\Phi_{1n} + b_{3n}\Phi_{2n}) - \mathbf{U}_m \cdot \nabla S_m^{(\Phi_1)}, \\ \dfrac{d}{dt} S_m^{(\Phi_2)} = \displaystyle\sum_{n=1}^{m} (c_{1n}\theta_n + c_{2n}\Phi_{1n} + c_{3n}\Phi_{2n}) - \mathbf{U}_m \cdot \nabla S_m^{(\Phi_2)}, \end{cases} \tag{44}$$

$$\theta_n = \Phi_{1n} = \Phi_{2n} = 0,\ \forall n \in \{1, ..., m\} \text{ for } z = 0, 1. \tag{45}$$

$$\theta_n(0) = \theta_n^{(0)},\ \Phi_{in}(0) = \Phi_{in}^{(0)},\ \ i = 1, 2, n \in \{1, ..., m\}. \tag{46}$$

$\forall m \in \mathbb{N}$, is guaranteed by (23). On the other hand, introducing the evolution system governing the n-th component of the of the perturbation (θ, Φ_1, Φ_2)

$$\begin{cases} \dfrac{\partial}{\partial t}\theta_n = a_{1n}\theta_n + a_{2n}\Phi_{1n} + a_{3n}\Phi_{2n} - \mathbf{U}_m \cdot \nabla\theta_n, \\ \dfrac{\partial}{\partial t}\Phi_{1n} = b_{1n}\theta_n + b_{2n}\Phi_{1n} + b_{3n}\Phi_{2n}) - \mathbf{U}_m \cdot \nabla\Phi_{1n}, \\ \dfrac{\partial}{\partial t}\Phi_{2n} = c_{1n}\theta_n + c_{2n}\Phi_{1n} + c_{3n}\Phi_{2n}) - \mathbf{U}_m \cdot \nabla\Phi_{2n}, \end{cases} \tag{47}$$

$$\begin{cases} \mathbf{U}_m = \sum_{n=1}^{m} \mathbf{u}_n, \mathbf{u}_n = \frac{1}{a^2}\left(\frac{\partial^2 \omega_n}{\partial x \partial z}\mathbf{i} + \frac{\partial^2 \omega_n}{\partial y \partial z}\mathbf{j} + \omega_n \mathbf{k}\right), \\ \omega_n = \tilde{\omega}_n(x,y,t)\sin(n\pi z), \theta_n = \tilde{\theta}_n(x,y,t)\sin(n\pi z), \\ \Phi_{in} = \tilde{\Phi}_{in}(x,y,t)\sin(n\pi z), i = 1,2, n \in \{1,...,m\} \end{cases} \tag{48}$$

(44) are immediately obtained by adding with respect to n from $n = 1$ to $n = m$ each equations of (47). Therefore the nonexistence of subcritical instabilities and the global stability is guaranteed by showing that (23) imply the asymptotic stability of the null solution of (45)-(48) $\forall m \in \mathbb{N}$.

Theorem 4. *Let (23) hold. Then, for any $m \in \mathbb{N}$, the zero solution of (45)-(48) is asymptotically stable for any initial data*

Proof. Denoting by λ_{ni}, $(i = 1, 2, 3)$, the roots of (21), (23) guarantee that λ_{ni} have negative real part and at least one - say λ_{n1} - be a negative real number. Then denoting by $\tilde{\mathbf{U}}_n = (1, \tilde{U}_{n2}, \tilde{U}_{n3})$ an eigenvector associated to λ_{n1}, Lemma 1 can be applied by setting

$$\begin{cases} Z_{n1} = \theta_n, Z_{n2} = \Phi_{1n} - \tilde{U}_{n2}\theta_n, Z_{n3} = \Phi_{2n} - \tilde{U}_{n3}\theta_n, \\ F_1 = \mathbf{U}_m \cdot \nabla\theta_n,\ F_2 = \mathbf{U}_m \cdot \nabla\Phi_{1n}, F_3 = \mathbf{U}_m \cdot \nabla\Phi_{2n}, \\ \tilde{F}_1 = F_1, \tilde{F}_2 = F_2 - \tilde{U}_{n2}F_1, \tilde{F}_3 = F_3 - \tilde{U}_{n3}F_1 \end{cases} \tag{49}$$

Let

$$\tilde{V}_n = \int_\Omega \tilde{W}_n d\Omega, \tag{50}$$

with $\tilde{W}_n$ given by (39) with λ_{ni} at the place of λ_i. Then - instead of (41) - one has to show that

$$\tilde{\Psi} =< \tilde{A}_{1n}Z_{n2} - \tilde{A}_{3n}Z_{n3}, \tilde{F}_2 > + < \tilde{A}_{2n}Z_{n3} - \tilde{A}_{3n}Z_{n2}, \tilde{F}_3 > + < Z_{n1}, \tilde{F}_1 >= 0, \tag{51}$$

with $\tilde{A}_{1n}, \tilde{A}_{2n}, \tilde{A}_{3n}$ constants.
In view of (49) it follows that Z_{ni}, $(i = 1, 2, 3))$, is of the kind

$$Z_{ni} = \tilde{Z}_{ni}(x, y, t)\sin(n\pi z), \tag{52}$$

$$\mathbf{U}_m \cdot \nabla\theta_n = \sum_{p=1}^{m}\left[\frac{p\pi}{a^2}\left(\frac{\partial\tilde{\omega}_p}{\partial x}\frac{\partial\tilde{\theta}_n}{\partial x} + \frac{\partial\tilde{\omega}_p}{\partial y}\frac{\partial\tilde{\theta}_p}{\partial x}\right)\cos(p\pi z)\sin(n\pi z) + \right.$$
$$\left. n\pi\tilde{\omega}_p\tilde{\theta}_p\sin(p\pi z)\cos(n\pi z)\right] \tag{53}$$

and hence

$$< Z_m, \tilde{F}_1 >=< \frac{\tilde{Z}_{n1}\sin(n\pi z)}{U^*_{n1}}, \mathbf{U}_m \cdot \nabla\theta_n > . \tag{54}$$

Since it easily turns out that

$$\int_0^1 \sin(q\pi z)\cos(p\pi z)\sin(n\pi z)dz = 0, \text{for } p+q \neq n. \tag{55}$$

But one easily verifies that all the other scalar products appearing in $\tilde{\Psi}$ are linear combinations of terms of kind (54) (with Φ_{in} at the place of θ_n) and hence by virtue of (55) it turn out that $\tilde{\Psi} = 0$.

6 Proof of (7)

In the case $P_1 = 1$, (4) reduces to

$$\begin{cases} \nabla p = -\mathbf{u} + (R\theta - R_1\Phi_1 - R_2\Phi_2)\mathbf{k}, \\ \theta_t = R\omega + \Delta\theta - \mathbf{u}\cdot\nabla\theta, \\ \Phi_{1t} = R_1\omega + \Delta\Phi_1 - \mathbf{u}\cdot\Phi_1, \\ \Phi_{2t} = -\dfrac{R_2}{P_2}\omega + \dfrac{1}{P_2}\Delta\Phi_1 - \dfrac{\mathbf{u}}{P_2}\cdot\Phi_2. \end{cases} \tag{56}$$

Setting

$$\varphi = R_1\theta - R\Phi_1 = 0 \Leftrightarrow \Phi_1 = \frac{1}{R}(R_1\theta - \varphi), \tag{57}$$

it follows that

$$\nabla p = -\mathbf{u} + \left(\frac{R^2 - R_1^2}{R}\theta + \frac{R_1}{R}\varphi - R_2\Phi_2\right)\mathbf{k}, \tag{58}$$

$$\begin{cases} \varphi_t = \Delta\varphi - \mathbf{u}\cdot\nabla\varphi, \\ \theta_t = R\omega + \Delta\theta - \mathbf{u}\cdot\nabla\theta, \\ \Phi_{2t} = -\dfrac{R_2}{P_2}\omega + \dfrac{1}{P_2}\Delta\Phi_2 - \dfrac{\mathbf{u}\cdot\nabla\Phi_2}{P_2}. \end{cases} \tag{59}$$

In view of theorem 1, one obtains

$$\tilde{\omega}_n = \eta_n\left(\frac{R^2 - R_1^2}{R}\tilde{\theta}_n + \frac{R_1}{R}\tilde{\varphi}_n - R_2\tilde{\Phi}_{2n}\right), \tag{60}$$

and hence

$$\begin{cases} \varphi_t = \displaystyle\sum_1^\infty (\bar{\alpha}_{1n}\varphi_n + \bar{\alpha}_{2n}\theta_n + \bar{\alpha}_{3n}\Phi_{2n}) - \mathbf{u}\cdot\nabla\varphi, \\ \theta_t = \displaystyle\sum_1^\infty \left(\bar{\beta}_{1n}\varphi_n + \bar{\beta}_{2n}\theta_n + \bar{\beta}_{3n}\Phi_{2n}\right) - \mathbf{u}\cdot\nabla\theta, \\ \Phi_{2t} = \displaystyle\sum_1^\infty (\bar{\gamma}_{1n}\varphi_n + \bar{\gamma}_{2n}\theta_n + \bar{\gamma}_{3n}\Phi_{2n}) - \mathbf{u}\cdot\nabla\Phi_2, \end{cases} \tag{61}$$

with

$$\begin{cases} \bar{\alpha}_{1n} = -\xi_n, \qquad \bar{\alpha}_{2n} = \bar{\alpha}_{3n} = 0, \\ \bar{\beta}_{1n} = R_1\eta_n, \bar{\beta}_{2n} = (R^2 - R_1^2)\eta_n - \xi_n, \bar{\beta}_{3n} = -RR_2\eta_n, \\ \bar{\gamma}_{1n} = -\dfrac{R_1R_2}{RP_2}\eta_n, \bar{\gamma}_{2n} = -\dfrac{R_2(R^2 - R_1^2)}{RP_2}\eta_n, \bar{\gamma}_{3n} = \dfrac{R_2^2\eta_n - \xi_n}{P_2}. \end{cases} \tag{62}$$

The auxiliary system, governing the $n-$th component of the fields $(\varphi, \theta, \Phi_{2n})$ analogous to (47), can easily found to be

$$\begin{cases} \dfrac{\partial}{\partial t}\varphi_n = -\xi_n\varphi_n + \ 0 \quad + \quad 0 \ - \mathbf{U}_m\cdot\nabla\varphi_n, \\ \dfrac{\partial}{\partial t}\theta_n = \bar{\beta}_{1n}\varphi_n + \bar{\beta}_{2n}\theta_n + \bar{\beta}_{3n}\Phi_{2n} - \mathbf{U}_m\cdot\nabla\theta_n, \\ \dfrac{\partial}{\partial t}\Phi_{2n} = \bar{\gamma}_{1n}\varphi_n + \bar{\gamma}_{2n}\theta_n + \bar{\gamma}_{3n}\Phi_{2n} - \mathbf{U}_m\cdot\nabla\Phi_{2n}. \end{cases} \tag{63}$$

Setting

$$\begin{cases} \mathcal{I}_n = \bar{\beta}_{2n} + \bar{\gamma}_{3n} = \eta_n\left[R^2 - R_1^2 + \dfrac{R_2^2}{P_2} - \left(1 + \dfrac{1}{P_2}\right)\dfrac{\xi_n^2}{a^2}\right], \\ \mathcal{A}_n = \bar{\beta}_{2n}\bar{\gamma}_{3n} + \bar{\beta}_{3n}\bar{\gamma}_{2n} = \eta_n\xi_n\left[-(R^2 - R_1^2) - R_2^2 - \dfrac{\xi_n^2}{a^2}\right], \end{cases} \tag{64}$$

$$\tilde{V}_n = \frac{1}{2}\int_\Omega \left\{\varphi_n^2 + \mathcal{A}_n(\theta_n^2 + \Phi_{2n}^2) + (\bar{\beta}_{2n}\Phi_{2n} - \bar{\gamma}_{2n}\theta_n)^2 + (\bar{\beta}_{3n}\Phi_{2n} - \bar{\gamma}_{2n}\theta_n)^2\right\} ds \tag{65}$$

it follows that

$$\frac{d\tilde{V}_n}{dt} \le \frac{1}{2}\int_\Omega \left[-\xi_n\varphi_n^2 + \mathcal{I}_n\mathcal{A}_n(\theta_n^2 + \Phi_{2n}^2)\right] d\Omega \tag{66}$$

with

$$\xi_n = a^2 + n^2\pi^2 \ge \pi^2. \tag{67}$$

Therefore

$$\begin{cases} \mathcal{I}_n < 0 \Leftrightarrow R^2 < R_1^2 - \dfrac{R_2^2}{P_2} + 4\pi^2\left(1 + \dfrac{1}{P_2}\right) \\ \\ \mathcal{A}_n > 0 \Leftrightarrow R^2 < R_1^2 - R_2^2 + 4\pi^2 \end{cases} \tag{68}$$

i.e. (7) guarantee the global stability.

Remark 3. We remark that:

1) The same procedure can be applied for obtaining (8)-(9) [29];

2) in the cases $P_2 \ge 1$, $P_1 \le 1$, (7)-(8) reduce to

$$R_C^2 = R_1^2 - R_2^2 + 4\pi^2. \tag{69}$$

Acknowledgements. This paper has been performed under the auspices of G.N.F.M. of I.N.D.A.M. and was supported in part from the Leverhulm Trust, "Tipping points: mathematics metaphors and meanings".

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